4.4b Simplification Orders and Recursive Path Orders (Part 2)

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Ex. 4.4.10 There are natural TRSs where LPOS fails.

$$plus(O, y) \to y$$

$$plus(\underline{s(x)}, \underline{y}) \to s(plus(\underline{y}, \underline{x}))$$

Comparing arguments from left to right doesn't work:

$$s(x) \not>_{lpos} y$$

Comparing arguments from right to left also doesn't work:

$$y \not>_{lpos} x$$

Solution: introduce another order where arguments are compared as multisets

- position of arguments is irrelevant
- the same argument may occur several times

$$\{1,1,4\} = \{1,4,1\} \neq \{1,4\}$$

Def 4.4.11 (Multiset)

Let T be a non-empty set. An unordered finite sequence M of elements from T is called a multiset over T. The set of all (finite) multisets is over T. The set of all (finite) multisets is denoted $\mathcal{M}(T)$.

A multiset M can be defined by its <u>characteristic function</u> $\#_M : T \to \mathbb{N}$, where $\#_M(t)$ is the number of times that t occurs in the multiset M. Thus: $t \in M$ iff $\#_M(t) > 0$.

We define the following operations on multisets:

- $M = N$ iff $\#_M(t) = \#_N(t)$ for all $t \in T$
- $M \subseteq N$ iff $\#_M(t) \leq \#_N(t)$ — " —
- $M \subset N$ iff $M \subseteq N$ and $M \neq N$
- $M \cup N$ is the multiset with $\#_{M \cup N}(t) = \#_M(t) + \#_N(t)$
- $M \cap N$ is the multiset with $\#_{M \cap N}(t) = \min(\#_M(t), \#_N(t))$
- $M \setminus N$ — " — $\#_{M \setminus N}(t) = \max(0, \#_M(t) - \#_N(t))$

<u>Ex 4.4.12</u>

Let $M, N \in \mathcal{M}(\mathbb{N})$ with

$M = \{1, 1, 4\}$, $N = \{1, 2, 2, 4, 4, 5\}$

- $M \nsubseteq N$, $N \nsubseteq M$

- $M \cup N = \{1,1,1,2,2,4,4,4,5\}$
- $M \cap N = \{1,4\}$
- $M \setminus N = \{1\}$
- $N \setminus M = \{2,2,4,5\}$

For LPO, we showed that any order $\succ$ can be extended to an order $\succ_{lex}$ on tuples by comparing components lexicographically.

Now we show that $\succ$ can also be extended to an order $\succ_{mul}$ on tuples by comparing them as multisets.

Idea for $\succ_{mul}$: $M \succ_{mul} N$ iff N results from M by replacing some elements in M by $\succ$-smaller elements.

Def 4.4.13 (Multiset Relation)

Let $\succ$ be a relation on T (i.e., $\succ \subseteq T \times T$).

Then $\succ_{mul} \subseteq \mathcal{M}(T) \times \mathcal{M}(T)$ is defined as follows:

$M \succ_{mul} N$ iff there exist $X, Y \in \mathcal{M}(T)$ with:

- $\emptyset \neq X \subseteq M$
- $N = (M \setminus X) \cup Y$
- for every $y \in Y$ there exists an $x \in X$ such that $x \succ y$

<u>Ex 4.4.14</u> $M = \{\underline{3}, 6, \underline{8}\}$ $N = \{\underline{4}, \underline{5}, 6, \underline{6}, \underline{7}, \underline{7}\}$

We have $M \; (>_{\mathbb{N}})_{mul} \; N$,

because we replaced

$X = \{3, 8\} \longleftarrow M \setminus N$

by $Y = \{4, 5, 6, 7, 7\} \longleftarrow N \setminus M$

Indeed, for every $y \in Y$, there is $8 \in X$ with $8 >_{\mathbb{N}} y$.

If $\succ$ is irreflexive, then we always have

$X = M \setminus N$

$Y = N \setminus M$

We showed that $\succ$ is well founded iff $\succ_{lex}$ is well founded.

Now we show the analogous observation for multisets.

<u>Thm 4.4.15</u> (Well-Foundedness of $\succ_{mul}$, Dershowitz+Manna 1979)

$\succ$ is well founded iff $\succ_{mul}$ is well founded.

<u>Proof</u>: "$\Leftarrow$": Let $\succ_{mul}$ be well founded.

Assume that $\succ$ is not well founded, i.e.,

$$t_0 \succ t_1 \succ t_2 \succ \dots$$

$$\rightsquigarrow \quad \{t_0\} \succ_{mul} \{t_1\} \succ_{mul} \{t_2\} \succ_{mul} \dots$$

$\lightning$ to the well-foundedness of $\succ_{mul}$.

"$\Rightarrow$": Let $\succ$ be well founded.

Assume that $\succ_{mul}$ is not well founded, i.e.,

$$M_0 \succ_{mul} M_1 \succ_{mul} M_2 \succ_{mul} \dots$$

From this infinite sequence, we construct an <u>infinite</u> tree:

- nodes are labeled with elements from T
- parents are $\succ$-greater than its children.

  The elements along a path are decreasing w.r.t. $\succ$.
- branching factor is <u>finite</u>

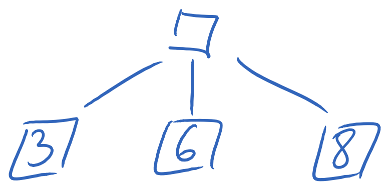
By Lemma of König (Thm. 4.1.3) then the tree must have an infinite path. Since elements on a path are decreasing w.r.t. $\succ$, this contradicts well-foundedness of $\succ$.

Now we have to explain how to construct this tree.
We allow an additional element $\perp$ ("bottom") as a possible label of nodes and we define $t \succ \perp$ for all $t \in T$.

Stepwise construction of the tree with the following invariant:
After construction step i, the leaves of the tree are M_i (and some more leaves may be labelled with $\perp$).

<u>Construction Step 0</u>: Non-labelled root node, its children are the elements of M_0.

Ex: $M_0 = \{3, 6, 8\}$



<u>Construction Step 1</u>:

$M_0 \succ_{mul} M_1$: There exist corresponding subsets X, Y as in Def 4.4.13.

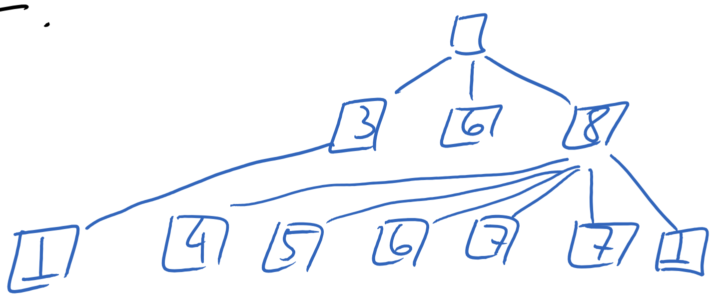
For every $y \in Y$ there exists an $x \in X$ with $x \succ y$.

- Add new leaves for all $y \in Y$.
- Parent of $y \in Y$ is a former leaf $x \in X$ with $x \succ y$.

- In addition, every former leaf $x \in X$ gets an additional child node marked with $\perp$.

$M_0 = \{\underline{3}, 6, \underline{8}\}$ $\quad M_1 = \{\underline{4}, \underline{5}, 6, \underline{6}, \underline{7}, \underline{7}\}$

$X = \{3, 8\}$ $\quad Y = \{4, 5, 6, 7, 7\}$



Now the leaves are labelled with M_1 (and $\perp$).

$\perp$ nodes are needed to ensure that $x \in X$ that have no $y \in Y$ with $x > y$ are no leaves anymore.

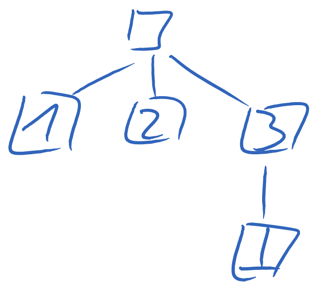
This construction is repeated infinitely many times.

- branching factor is finite (since all M_i are finite)
- number of nodes increases (by at least one $\perp$-node) in each step
- elements are decreasing w.r.t. $>$ on each path

↯ to well-foundedness of $>$ ☒

$M_0 = \{1, 2, 3\}$ $\quad M_1 = \{1, 2\}$

$X = \{3\}$ $\quad Y = \emptyset$



$M_2 = \{1, 1, 1, 1, 1, 1\}$

Def 4.4.16 (Recursive Path Order, Dershowitz 1982)

Let $\sqsupset$ be a well-founded order on Σ ("precedence").

The *recursive path order* (RPO) on $\mathcal{T}(\Sigma, \mathcal{V})$ is defined as $s \succ_{rpo} t$ iff

- $s = f(s_1, \dots, s_n)$ and $s_i \succeq_{rpo} t$ for some $i \in \{1, \dots, n\}$ or
- $s = f(s_1, \dots, s_n)$, $t = g(t_1, \dots, t_m)$, $f \sqsupset g$, and $s \succ_{rpo} t_j$ for all $1 \le j \le m$ or
- $s = f(s_1, \dots, s_n)$, $t = f(t_1, \dots, t_n)$, and $\{s_1, \dots, s_n\} \ (\succ_{rpo})_{mul} \ \{t_1, \dots, t_n\}$

Ex 4.4.17

$$plus(0, y) \to y$$
$$plus(s(x), y) \to s(plus(y, x))$$

Now we have:

$plus(0, y) \succ_{rpo} y$

$plus(s(x), y) \succ_{rpo} s(plus(y, x))$, ← requires $plus \sqsupset succ$

Since $plus(s(x), y) \succ_{rpo} plus(y, x)$,

Since $\{s(x), y\} \ (\succ_{rpo})_{mul} \ \{y, x\}$,

since $s(x) \succ_{rpo} x$

RPO can also be used for termination proofs, since it is a reduction order.

Thm 4.4.18 (Properties of RPO)

$\succ_{rpo}$ is a simplification order.

Proof: similar as for $\succ_{lpo}$ (Thm 4.4.9)

LPOS and RPO can be combined.

$$
\begin{aligned}
&plus(0, y) \to y \\
&plus(s(x), y) \to s(plus(y, x)) \leftarrow \text{LPOS fails} \\
&sum(0, y) \to y \\
&sum(s(x), y) \to sum(x, s(y)) \leftarrow \text{RPO fails}
\end{aligned}
$$

Solution: RPOS, i.e., status can also be "multiset".

Here: status of sum : $\langle 1, 2 \rangle$

status of plus : "multiset"

Question whether there is an RPOS that orients all rules of a TRS is decidable (and NP-complete).